Permutations

Permutations represent the number of different possible ways we can **arrange** a number of elements.

$$P(n) = n \times (n-1) \times (n-2) \times \cdots \times 1$$

Permutations

Options for who we put first

Options for who we put second

Options for who we put last

Characteristics of Permutations:

- Arranging **all** elements within the sample space.
- No repetition.
- $P(n) = n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$ (Called "n factorial")

Example:

- If we need to arrange 5 people, we would have P(5) = 120 ways of doing so.

Factorials

Factorials express the **product** of all integers from **1** to **n** and we denote them with the "!" symbol.

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

Key Values:

- 0! = 1.
- If n<0, n! does not exist.

Rules for factorial multiplication. (For n>0 and n>k)

- $(n+k)! = n! \times (n+1) \times \cdots \times (n+k)$
- $(n-k)! = \frac{n!}{(n-k+1)\times\cdots\times(n-k+k)} = \frac{n!}{(n-k+1)\times\cdots\times n}$
- $\frac{n!}{k!} = \frac{k!\times(k+1)\times\cdots\times n}{k!} = (k+1)\times\cdots\times n$

Examples: n = 7, k = 4

- $(7+4)! = 11! = 7! \times 8 \times 9 \times 10 \times 11$
- $(7-4)! = 3! = \frac{7!}{4\times5\times6\times\times7}$
- $\frac{7!}{4!} = 5 \times 6 \times 7$

Variations

Variations represent the number of different possible ways we can **pick** and **arrange** a number of elements.

Variations **with** repetition

$$\bar{V}(n,p) = n^p$$

Number of different elements available (n)

Number of elements we are arranging (p)

Variations without repetition

$$V(n,p) = \frac{n!}{(n-p)!}$$

Number of different elements available (n)

Number of elements we are arranging (p)

Intuition behind the formula. (With Repetition)

- We have n-many options for the first element.
- We **still have n-many options** for the second element because repetition is allowed.
- We have n-many options for each of the p-many elements.
- $n \times n \times n \ldots n = n^p$

Intuition behind the formula. (Without Repetition)

- We have n-many options for the first element.
- We **only have (n-1)-many options** for the second element because we cannot repeat the value for we chose to start with.
- We have less options left for each additional element.
- $n \times (n-1) \times (n-2) \ldots (n-p+1) = \frac{n!}{(n-p)!}$

Combinations

Combinations represent the number of different possible ways we can **pick** a number of elements.

$$C(n,p) = \frac{n!}{(n-p)!\,p!}$$

Combinations

Total number of elements in the sample space

Number of elements we need to select

Characteristics of Combinations:

- Takes into account double-counting. (Selecting Johny, Kate and Marie is the same as selecting Marie, Kate and Johny)
- All the different permutations of a single combination are different variations.
- $C = \frac{V}{P} = \frac{n!/(n-p)!}{p!} = \frac{n!}{(n-p)!p!}$
- Combinations are symmetric, so $C_p^n = C_{n-p}^n$, since selecting p elements is the same as omitting n-p elements.

Combinations with separate sample spaces

Combinations represent the number of different possible ways we can **pick** a number of elements.

$$C = n_1 \times n_2 \times \cdots \times n_p$$

Combinations

Size of the first sample space.

Size of the second sample space.

Size of the last sample space.

Characteristics of Combinations with separate sample spaces:

- The option we choose for any element does not affect the number of options for the other elements.
- The order in which we pick the individual elements is arbitrary.
- We need to know the size of the sample space for each individual element. $(n_1, n_2 \ldots n_p)$

Winning the Lottery

To win the lottery, you need to satisfy two distinct independent events:

- Correctly guess the "Powerball" number. (From 1 to 26)
- Correctly guess the 5 regular numbers. (From 1 to 69)

Total number of Combinations

$$C = \underbrace{\frac{69!}{64!\,5!}}_{C_{5\ numbers}} \times \underbrace{26}_{C_{Powerball\ number}}$$

Intuition behind the formula:

- We consider the two distinct events as a combination of two elements with different sample sizes.
  - One event has a sample size of 26, the other has a sample size of C_5^{69}.
- Using the "favoured over all" formula, we find the probability of any single ticket winning equals $1/(\frac{69!}{64!5!} \times 26)$.